Lezione 10

7.12: 7.19

7.20:

7.12: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ Trova base ortogonale $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ v ortogonale a e_1 $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$(1\ 0)\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ $\quad x + y = 0$ $\quad$ Es: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ ok

Cerco un $\begin{pmatrix} x \\ y \end{pmatrix}$ non isotropo:

$\begin{pmatrix} \boxed{0} & -1 \\ -1 & \boxed{0} \end{pmatrix}$ $\quad (x\ y)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \neq 0$

$\Bigg(0\ 0\ 1 \Bigg)$

$$(-y \;\; -x)\begin{pmatrix} x \\ y \end{pmatrix} \neq 0 \qquad -2xy \neq 0$$

Es: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$ non è isotropo

Cerco v_2 ortogonale a v_1: $(1 \;\; 1)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$(-1 \;\; -1)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad -x-y=0 \qquad \text{Es: } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ è ortogonale

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

e_2 non è isotropo $\Rightarrow$ $g|_U$ nondeg $\Rightarrow U \oplus U^\perp = \mathbb{R}^3$

$\underline{U = \mathrm{Span}(e_2)}$

$$\boxed{U^\perp \oplus U = \mathbb{R}^3}$$

continuare con $U^\perp$

$$U^\perp = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \underbrace{(0\,1\,0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$

$$U^\perp = \left\{ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\} = \{ y = 0 \}$$

Cerchiamo base ortogonale per $U^\perp$

Cerco un vettore non isotropo di $U^\perp = \{y=0\} = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \right\}$

$$\begin{pmatrix} x & 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \neq 0$$

$$\begin{pmatrix} z & 0 & x \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = 2xz \neq 0 \iff x \neq 0 \wedge z \neq 0$$

Es: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$v_1 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Cerco $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U^\perp$ (cioè $y=0$)

$$v_3 = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$$

che sia ortogonale a v_2: $\quad \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = 0$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = x + z = 0$$

Esempio: $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\qquad \mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

base ortogonale

7.19:

Determina g t.c. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ siano base ortonormale

$$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v_2} \right\} \qquad [g]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{B}$ ortonormale $\iff [g]_{\mathcal{B}} = I$

$[g]_e = ?$

$$[g]_{\mathcal{C}} = {}^t[id]^{\mathcal{C}}_{\mathcal{B}}\,[g]_{\mathcal{B}}\,\underset{\substack{\| \\ M}}{[id]^{\mathcal{C}}_{\mathcal{B}}}$$

$$M = [id]^{\mathcal{C}}_{\mathcal{B}}$$

$$M^{-1} = [id]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = -1 \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow [g]_{\mathcal{C}} = {}^tM\,[g]_{\mathcal{B}}\,M$$

$$= \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = S$$

$S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ Verifica: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ e $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ sono base ortonormale?

$$g_S\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 1 \qquad g_S\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 0$$

VERIFICHE NON NECESSARIE

$$g_S\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 1$$

$$(1 \ \ 1)\underbrace{\begin{pmatrix} +2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = (1 \ \ 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

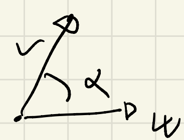
~~non~~ torna

7.20

PRODOTTI SCALARI DEFINITI POSITIVI

V con prod. scalare def+

- $\|v\| \geq 0$ NORMA DI UN VETTORE $v \in V$
- $d(P, Q)$ DISTANZA FRA PUNTI $P, Q \in V$
- angolo fra due vettori non nulli

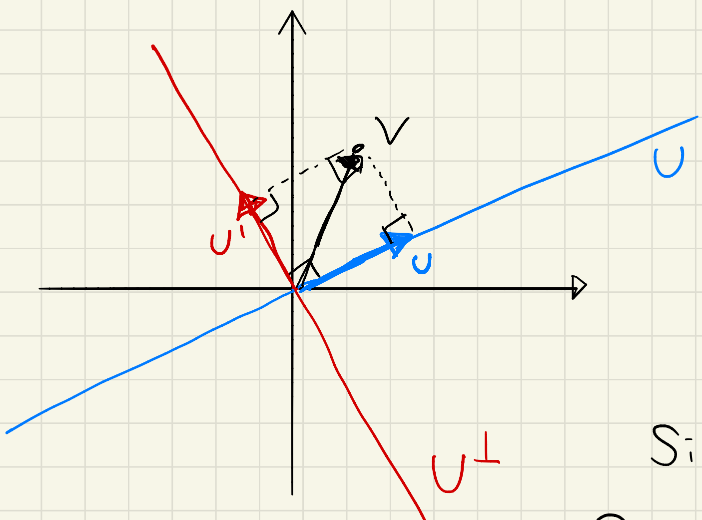


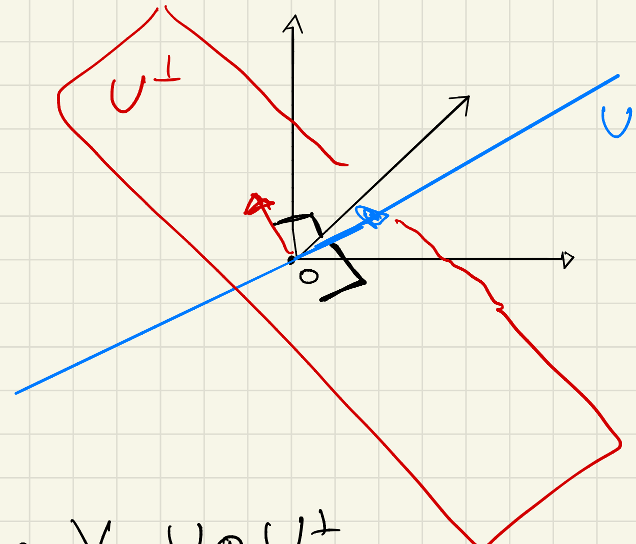
Es: $\mathbb{R}^n$ con prod. scalare Euclideo.

PROIEZIONE ORTOGONALE

$U \subseteq V$ sottospazio $\Rightarrow$ $V = U \oplus U^\perp$ perché g è def +

Se $v \in V$





Siccome $V = U \oplus U^\perp$

Ogni $v \in V$ si scrive in modo UNICO

come $v = u + u'$ con $u \in U$, $u' \in U^\perp$

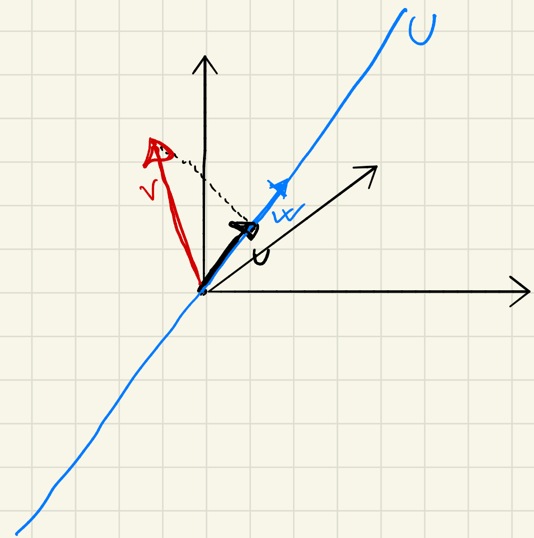
Chiamo u la PROIEZIONE ORTOGONALE di v su U

$$\boxed{u = P_U(v)}$$

$P_U : V \to V$ endomorfismo che manda v sulla sua proiezione u in U

Come calcolare u?

Se $U = \mathrm{Span}(w)$ è facile:



In questo caso

$$P_w(v) = P_U(v) = \frac{\langle v,w\rangle}{\langle w,w\rangle} w$$

COEFFICIENTE DI FOURIER

Prop: La formula funziona.

dim: Cerco un numero λ t.c. $P_w(v) = \lambda w$

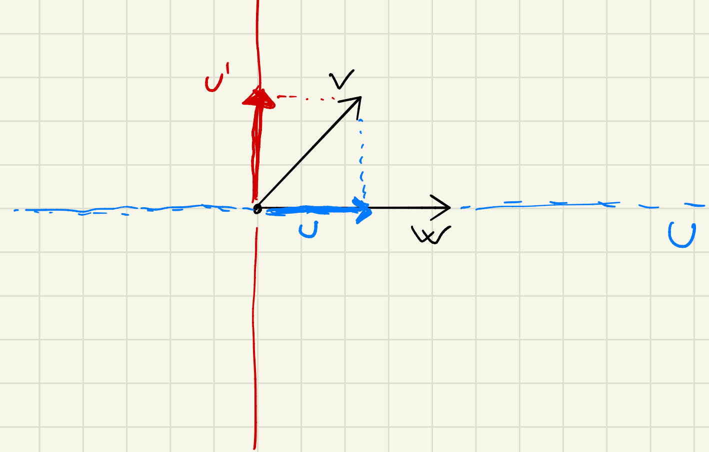
Quale condizione impongo?

$u = \lambda w$ $\quad\quad$ λ incognita

$u' = v - u \Leftarrow v = u + u'$

deve venire ortogonale a w

cioè $\quad \langle u', w \rangle = 0$



cioè $\quad \langle v - u, w \rangle = 0$

$$\langle v - \lambda w, w \rangle = 0 \iff \langle v, w \rangle - \lambda \langle w, w \rangle = 0$$

$$\iff \langle v, w \rangle = \lambda \langle w, w \rangle \iff \lambda = \frac{\langle v, w \rangle}{\langle w, w \rangle}$$

Esempi:

Es: $\quad w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ $\quad$ Calcola $P_w(v)$ $\quad$ $\mathbb{R}^3$ Euclideo

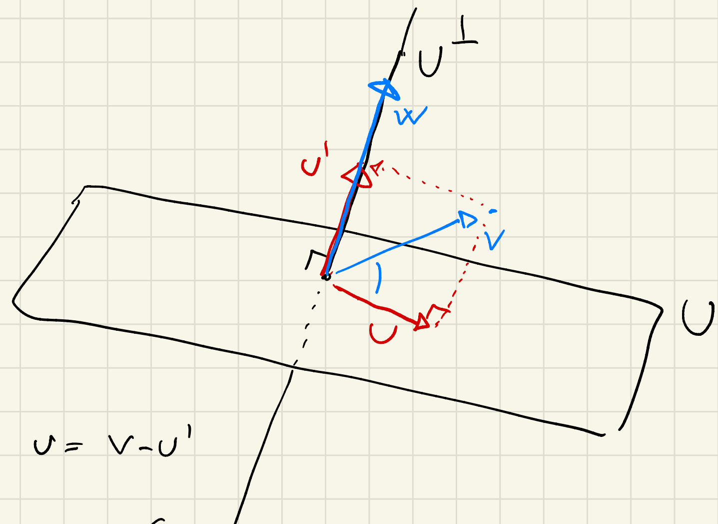
$$P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w = \frac{3 - 2 + 12}{1 + 4 + 9} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

$$= \frac{13}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Calcolo di proiezione ortogonale su un piano:

U piano in $\mathbb{R}^3$

Come calcolare $p_U(v)$?



Tecnica:

Calcolare $u' = p_{U^\perp}(v)$

e quindi $v = u + u' \Rightarrow u = v - u'$

Esempio:

$$v = \begin{pmatrix}2\\0\\5\end{pmatrix} \qquad U = \{x - y + z = 0\}$$

Trovo $u = p_U(v)$. Calcolo $u' = p_{U^\perp}(v)$:

$U^\perp$ è retta $U^\perp = \text{Span}(w)$ $w = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$$u' = p_w(v) = \frac{\langle v,w\rangle}{\langle w,w\rangle} w = \frac{2+0+5}{1+1+1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{7}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

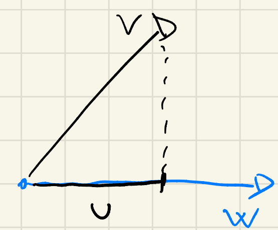
$$u = v - u' = \begin{pmatrix}2\\0\\5\end{pmatrix} - \frac{7}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1/3\\7/3\\8/3\end{pmatrix}$$

Verifica: 1) $u \in U$ 2) $\langle u, u'\rangle = 0$

OK

Es: $u = p_w(v)$

$$\|u\| = \|p_w(v)\| = \frac{|\langle v,w\rangle|}{\|w\|}$$



(è equivalente a dire $|\langle v,w\rangle| = \|u\|\cdot\|w\|$)

$$\|u\| = \|p_w(v)\| = \Bigg\|\underbrace{\frac{\langle v,w\rangle}{\langle w,w\rangle}}_{\text{scalare}} \underbrace{w}_{\text{vettore}}\Bigg\| = \left|\frac{\langle v,w\rangle}{\langle w,w\rangle}\right|\cdot\|w\| = \frac{|\langle v,w\rangle|}{|\langle w,w\rangle|}\|w\|$$

$$\|\lambda w\| = |\lambda| \cdot \|w\| \quad \text{in generale}$$

$$= \frac{|\langle v, w\rangle|}{\|w\|^{\cancel{2}}} \cdot \cancel{\|w\|} = \frac{|\langle v, w\rangle|}{\|w\|}$$

$$\boxed{P_w(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w}$$

<u>Prop</u>: $\{v_1, \dots, v_n\}$ base ortogonale per V

$v \in V$ generico. Come calcolo $v = \lambda_1 v_1 + \dots + \lambda_n v_n$

Generalmente i λ_i si trovano scrivendo un sistema lineare

In questo caso

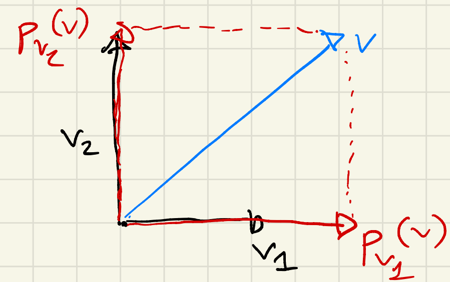
$$\boxed{\lambda_i = \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle}}$$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \lambda_1 \underline{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} + \lambda_2 \underline{\begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}} + \lambda_3 \underline{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

Troviamo $\lambda_1, \lambda_2, \lambda_3$ risolvendo sistema

$$v = P_{v_1}(v) + P_{v_2}(v)$$

$$v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$



PRODOTTO VETTORIALE

Es: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$

$\qquad\quad v_1 \qquad\quad v_2 \qquad\quad v_3$

$$(1\ 1\ 0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad (1\ {-1}\ 1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x + y = 0 \qquad x - y + z = 0$$

$$\ldots \begin{cases} x + y = 0 \\ x - y + z = 0 \end{cases}$$

$\times \quad \wedge \qquad v_1 \times v_2$ è ortogonale sia a v_1 che a v_2

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

Scrivere le coordinate di $\begin{pmatrix}x\\y\\z\end{pmatrix}$ rispetto a questa base.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{\left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle} \begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{\left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle} \begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{\left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\-1\\-2\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix} \right\rangle} \begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

$$= \frac{x+y}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{x-y+z}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{x-y-2z}{6}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

<u>Oss</u>: Se $\mathcal{B} = \{v_1, \ldots, v_n\}$ è ORTONORMALE

$\langle v_i, v_i \rangle = 1$ quindi $\dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} = \langle v, v_i \rangle$

Per qualsiasi $v \in V$ si ottiene

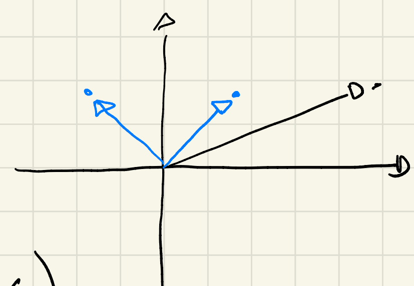
$$v = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \langle v, v_3\rangle v_3 + \dots + \langle v, v_n\rangle v_n$$

Esempio: $\{e_1, e_2, \dots, e_n\} = \mathcal{E}$ è ortonormale ($\mathbb{R}^n$ euclideo)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1}_{\langle x, e_1\rangle} e_1 + \underbrace{x_2}_{\langle x, e_2\rangle} e_2 + \dots + x_n e_n$$

$$v_1 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \qquad \mathbb{R}^2$$

$$\|v_1\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$



$$\underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{v} = \underbrace{\langle v, v_1\rangle}\, v_1 + \langle v, v_2\rangle v_2$$

$$= \frac{5\sqrt{2}}{2} \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} - \frac{\sqrt{2}}{2} \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$